Chapter 2. Small Oscillations

(Most of the material presented in this chapter is taken from Thornton and Marion, Chap. 3)

2.1 Introduction

If a particle, originally in a position of equilibrium (we limit ourselves to the case of motions in one dimension), is displaced by a small amount, a force will tend to bring it back to its original position. We assume that this restoring force F is only a function of position, i.e., $F = F(x)$. It is then easily expanded in a Taylor series:

$$F(x) = F_0 + x\left(\frac{dF}{dx}\right)_0 + \frac{1}{2!}x^2\left(\frac{d^2F}{dx^2}\right)_0 + \frac{1}{3!}x^3\left(\frac{d^3F}{dx^3}\right)_0 + \ldots \tag{2.1}$$

where F_0 is the value of $F(x)$ at the origin (i.e., position of equilibrium), and $\left(d^nF/dx^n\right)_0$ the nth derivative at the origin. Since the origin is defined to be the equilibrium point, we have $F_0 \equiv 0$. Furthermore, as **<u>we consider only small displacements</u>** from the origin, we will neglect all terms of second and higher powers of x. We can then rewrite equation (2.1) with

$$F = -kx \tag{2.2}$$

where $k \equiv -\left(dF/dx\right)_0$. (Strictly speaking, we should use '$\simeq$' instead of '='.) Because F is a restoring force (i.e., it brings the particle back toward its origin), its first derivative is negative and, therefore, k is positive. Systems that can be described with equation (2.2) obey **Hooke's Law** (the restoring force is approximately linear).

2.1 The Simple Harmonic Oscillator

If substitute Hooke's Law (equation (2.2)) into the Newtonian equation of motion $F = ma$, we get

$$\boxed{\ddot{x} + \omega_0^2 x = 0} \tag{2.3}$$

where we have defined a new quantity

$$\omega_0^2 \equiv \frac{k}{m}. \tag{2.4}$$

The solution to equation (2.3) can be expressed with sinusoidal functions such as

$$x(t) = A\sin(\omega_0 t - \delta) \tag{2.5}$$

or

$$x(t) = A\cos(\omega_0 t - \phi). \tag{2.6}$$

The constants A, δ (or ϕ) are determined by the initial conditions of the problem. For example, if at $t = 0$ the particle is located at $x(t = 0) \equiv x_0$, and moving with a velocity $\dot{x}(t = 0) \equiv \dot{x}_0$, we find that (if we choose equation (2.5) as the solution)

$$\begin{aligned} x_0 &= -A\sin(\delta) \\ \dot{x}_0 &= A\omega_0 \cos(\delta). \end{aligned} \tag{2.7}$$

Equation (2.7) is a system of two equations with two unknown that is readily solved, and gives

$$\begin{aligned} &\tan(\delta) = -\omega_0 \frac{x_0}{\dot{x}_0} \\ &A = -\frac{x_0}{\sin(\delta)}, \end{aligned} \tag{2.8}$$

with $x(t)$ given by equation (2.5). Incidentally, we can now appreciate that ω_0 is the **angular frequency** of the motion. It is related to frequency ν_0 by

$$\boxed{\begin{aligned} &\omega_0 = 2\pi\nu_0 = \sqrt{\frac{k}{m}} \\ &\nu_0 = \frac{1}{\tau_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \end{aligned}} \tag{2.9}$$

where τ_0 is the period. Note that these three quantities are independent of the amplitude A of the motion.

Alternatively, we can express A as a function of the total energy $E = T + U$ of the system. The kinetic energy T is given by

$$\begin{aligned} T = \frac{1}{2}m\dot{x}^2 &= \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega_0 t - \delta) \\ &= \frac{1}{2}kA^2 \cos^2(\omega_0 t - \delta), \end{aligned} \tag{2.10}$$

where we used equation (2.4) in the last step. The potential energy U is calculated from the work done in moving the particle a distance x. We start by calculating the incremental work dW necessary to move the particle by an amount dx <u>against</u> the force F (see equation (1.23) of chapter 1)

$$dW = -F\,dx = kx\,dx. \tag{2.11}$$

We now calculate

$$\begin{aligned} U &= \int_0^x dW \\ &= \int_0^x kx'dx' = \frac{1}{2}kx^2 \\ &= \frac{1}{2}kA^2 \sin^2\left(\omega_0 t - \delta\right). \end{aligned} \tag{2.12}$$

We finally get the total energy E

$$E = T + U = \frac{1}{2}kA^2. \tag{2.13}$$

We get the general result that the total energy of the system is proportional to *the square of the amplitude* of the motion. It is also independent of time (we assumed that the system exhibits no losses), and of the phase constant δ.

Example

Find the angular velocity and period of oscillation of a sphere of mass m and radius R about a point on its surface.

Solution. We assume that the sphere is homogeneous, with a constant mass density ρ. The moment of inertia of an object about a given axis is defined as the integral of its mass density multiplied by the square of its distance to the axis. The moment of inertia I_c of the sphere about an axis (we call it the z-axis) passing through its center is, therefore, given by (using the cylindrical coordinates r, φ, and z)

$$
\begin{aligned}
I_c &= \int_{-R}^{R}\int_{0}^{2\pi}\int_{0}^{\sqrt{R^2-z^2}} \rho r^2\, r dr\, d\varphi\, dz \\
&= 2\pi\rho \int_{-R}^{R}\int_{0}^{\sqrt{R^2-z^2}} r^3\, dr\, dz \\
&= 2\pi\rho \int_{-R}^{R} \frac{1}{4}\left(R^2 - z^2\right)^2 dz = \frac{\pi}{2}\rho \int_{-R}^{R}\left(R^4 - 2R^2z^2 + z^4\right) dz \\
&= \frac{\pi}{2}\rho\left(R^4 z - \frac{2}{3}R^2 z^3 + \frac{z^5}{5}\right)\Bigg|_{-R}^{R} \\
&= \pi\rho R^5\left(1 - \frac{2}{3} + \frac{1}{5}\right) \\
&= \frac{8}{15}\pi\rho R^5
\end{aligned}
$$

But since the mass density of the sphere (i.e., its mass divided by its volume) is given by $\rho = 3m/4\pi R^3$, we find

$$
I_c = \frac{2}{5} mR^2.
$$

It also turns out that the moment of inertia I of a sphere about an axis displaced by a distance R_d from its center is simply given by

$$
I = mR_d^2 + I_c.
$$

Since in our case $R_d = R$, we find that

$$
\begin{aligned}
I &= mR^2 + \frac{2}{5} mR^2 \\
&= \frac{7}{5} mR^2.
\end{aligned}
$$

If the sphere is subjected to gravity, and allowed to rotate about this (pivot) axis, we can calculate the torque applied to it by $\mathbf{N} = \mathbf{R} \times \mathbf{F}_g$, where $\mathbf{F}_g = m\mathbf{g}$ the gravitational force. If the angle of rotation of the sphere about the pivot axis is θ, and its angular acceleration $\ddot{\theta}$, we can write the equation of (angular) motion (remember equation (1.22) of chapter 1)

$$
I\ddot{\theta} = -Rmg\sin(\theta).
$$

However, if we simplify the problem by limiting ourselves to small oscillations, we can approximate $\sin(\theta) \simeq \theta$, and we have

$$\ddot{\theta} + \frac{mgR}{I}\theta = 0.$$

This is an equation similar to equation (2.3), and we, therefore, identify the angular frequency and period of oscillation of the sphere as

$$\omega_0 = \sqrt{\frac{mgR}{I}} = \sqrt{\frac{5g}{7R}}$$

and

$$\tau_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{7R}{5g}}.$$

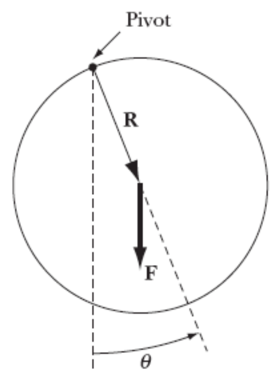


Figure 2.1 - A homogeneous sphere rotating about an (pivot) axis.

2.1 Harmonic Oscillations in Two Dimensions

We generalize the problem to allow motions with two degrees of freedom, or in two dimensions. The restoring force is now expressed as a vector, i.e., $\mathbf{F} = -k\mathbf{r}$. We can use a polar system of coordinates for the problem

$$\begin{aligned} F_x &= -kr\cos(\theta) = -kx \\ F_y &= -kr\sin(\theta) = -ky. \end{aligned} \tag{2.14}$$

The equations of motion are

$$\ddot{x} + \omega_0^2 x = 0$$
$$\ddot{y} + \omega_0^2 y = 0, \tag{2.15}$$

with $\omega_0^2 = k/m$. The solutions are similar to the case of the simple harmonic oscillator

$$x(t) = A\cos(\omega_0 t - \alpha)$$
$$y(t) = B\cos(\omega_0 t - \beta). \tag{2.16}$$

(Take note that we could have chosen sine instead of cosine functions.) Let us find an equation that expresses one coordinate as a function of the other. Consider the following

$$\frac{x^2(t)}{A^2} + \frac{y^2(t)}{B^2} = \cos^2(\omega_0 t - \alpha) + \cos^2(\omega_0 t - \beta). \tag{2.17}$$

But since $\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a-b) + \cos(a+b)\right)$, we can rewrite equation (2.17) as

$$\frac{x^2(t)}{A^2} + \frac{y^2(t)}{B^2} = 1 + \frac{1}{2}\left[\cos\left(2[\omega_0 t - \alpha]\right) + \cos\left(2[\omega_0 t - \beta]\right)\right]. \tag{2.18}$$

In a similar fashion, if we define

$$\delta \equiv [\omega_0 t - \beta] - [\omega_0 t - \alpha] = \alpha - \beta$$
$$\sigma \equiv [\omega_0 t - \beta] + [\omega_0 t - \alpha] = 2\omega_0 t - (\alpha + \beta).$$

we now have

$$\frac{x^2(t)}{A^2} + \frac{y^2(t)}{B^2} = 1 + \cos(\delta)\cos(\sigma). \tag{2.19}$$

Furthermore, since

$$\cos(\sigma) = 2\cos(\omega_0 t - \alpha)\cos(\omega_0 t - \beta) - \cos(\delta)$$
$$= 2\frac{x(t)}{A}\frac{y(t)}{B} - \cos(\delta), \tag{2.20}$$

we get

$$\frac{x^2(t)}{A^2} + \frac{y^2(t)}{B^2} = 1 + \cos(\delta)\left[2\frac{x(t)}{A}\frac{y(t)}{B} - \cos(\delta)\right], \tag{2.21}$$

or, after some rearrangement

$$\left(\frac{x(t)}{A}-\cos(\delta)\frac{y(t)}{B}\right)^2+\sin^2(\delta)\frac{y^2(t)}{B^2}=\sin^2(\delta). \tag{2.22}$$

We see that if the two oscillations (i.e., along the *x* and *y* axes) are out of phase by $\pm\pi/2$ (or, $\delta=\pm\pi/2$), equation (2.22) is that of an ellipse

$$\frac{x^2(t)}{A^2}+\frac{y^2(t)}{B^2}=1. \tag{2.23}$$

The special case when the amplitudes are equal (i.e., $A=B$) is that of a circular motion. On the other hand, if $\delta=0$ (which means that $\alpha=\beta$) we find that

$$\left(\frac{x(t)}{A}-\frac{y(t)}{B}\right)^2=0. \tag{2.24}$$

which is the equation of a straight line since it can be rewritten as

$$y(t)=\frac{B}{A}x(t), \qquad \delta=0. \tag{2.25}$$

This result could have been simply arrived at by setting $\alpha=\beta$ in equations (2.16). We could proceed in the same manner for the case where $\delta=\pm\pi$, which also yields the equation of a straight line but with opposite slope, i.e.,

$$y(t)=-\frac{B}{A}x(t), \qquad \delta=\pm\pi. \tag{2.26}$$

Equation (2.22) can be used to draw the motion of the particle in the $(x,\, y)$-plane, as is shown in the following figure for cases where $A=B$

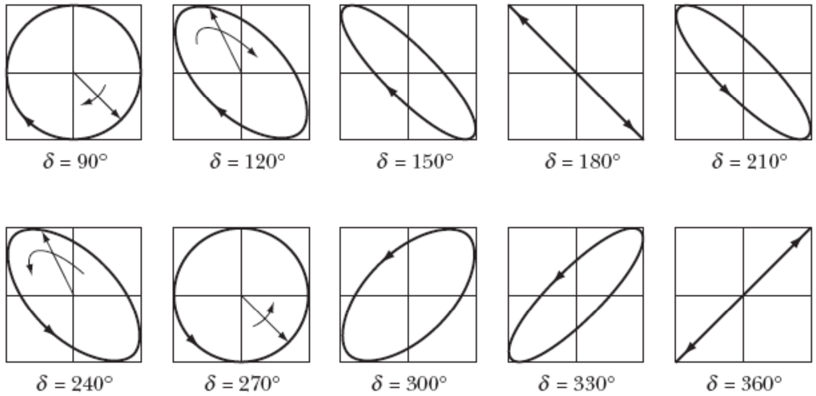


Figure 2.2 – Two-dimensional harmonic oscillation motion curves for different phase differences $\delta = \alpha - \beta$, when $A = B$.

In the more general case where the frequency of oscillation is different in the *x* and *y* directions, the solution becomes

$$\begin{aligned} x(t) &= A\cos(\omega_x t - \alpha) \\ y(t) &= B\cos(\omega_y t - \beta). \end{aligned} \tag{2.27}$$

The path of the motion in the $(x,\ y)$-plane is no longer an ellipse but a **Lissajous curve**. Such a motion can be *closed* or *open* depending on whether the ration ω_x/ω_y is a rational fraction or not. Take note that the shape of a curve not only depends on the phase difference $\delta \equiv \alpha - \beta$, but also on the individual angles α and β. Three examples are shown in this figure

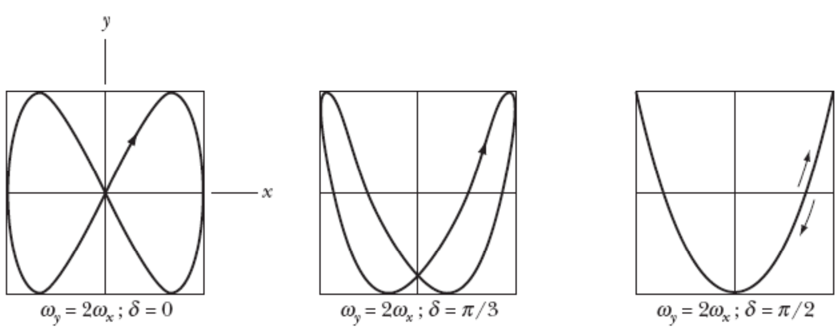
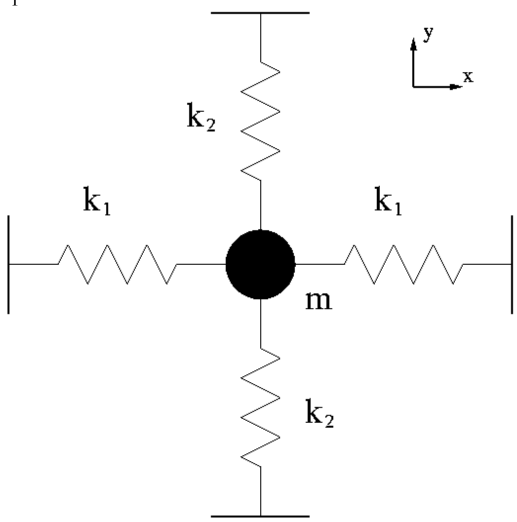


Figure 2.3 - Closed Lissajous curves. In these cases $\omega_y = 2\omega_x$, and $\delta = 0$ (i.e., $\alpha = \beta = \pi/2$), $\pi/3$ (i.e., $\alpha = 11\pi/6$ and $\beta = 3\pi/2$), and $\pi/2$ (i.e., $\alpha = 3\pi/2$ and $\beta = \pi$), respectively. The resulting curves depend strongly not only on δ, but also on α and β.

Example

An object of mass *m* is tied to four springs, two of constant k_1 in the *x* direction, and two others of constant k_2 in the *y* direction, which are in turn attached to walls. Assuming that we are dealing only with small oscillations, and that the initial conditions are x_0, y_0, and $\dot{x}_0 = \dot{y}_0 = 0$, find the equations that give the position of the object along both axes as a function if time. What kind of trajectory does the object take in the (x, y)-plane if $x_0 = y_0 = 1$, and $\omega_2 = 2\omega_1$?



Solution. The equations of motion for the object are

$$m\ddot{x} = -2k_1 x$$
$$m\ddot{y} = -2k_2 y.$$

Following a treatment similar to what was done for equations (2.15) and (2.16) we can write the following for the position of the mass

$$x(t) = A\cos(\omega_1 t - \alpha)$$
$$y(t) = B\cos(\omega_2 t - \beta),$$

where

$$\omega_1^{\,2} = \frac{2k_1}{m}$$
$$\omega_2^{\,2} = \frac{2k_2}{m}.$$

From the initial conditions defined for the problem we have

$$\begin{aligned} x_0 &= A\cos(\alpha) \\ y_0 &= B\cos(\beta) \\ 0 &= A\omega_1 \sin(\alpha) \\ 0 &= B\omega_2 \sin(\beta). \end{aligned}$$

The last two equations imply that α and β both are equal to either 0 or π. The first pair of equations can, however, be interpreted such that $\alpha = \beta = 0$, $x_0 = A$, and $y_0 = B$. We use the equations for $x(t)$ and $y(t)$ to determine the trajectory of the object in the $(x,\ y)$-plane. The result is shown in Figure 2.4.

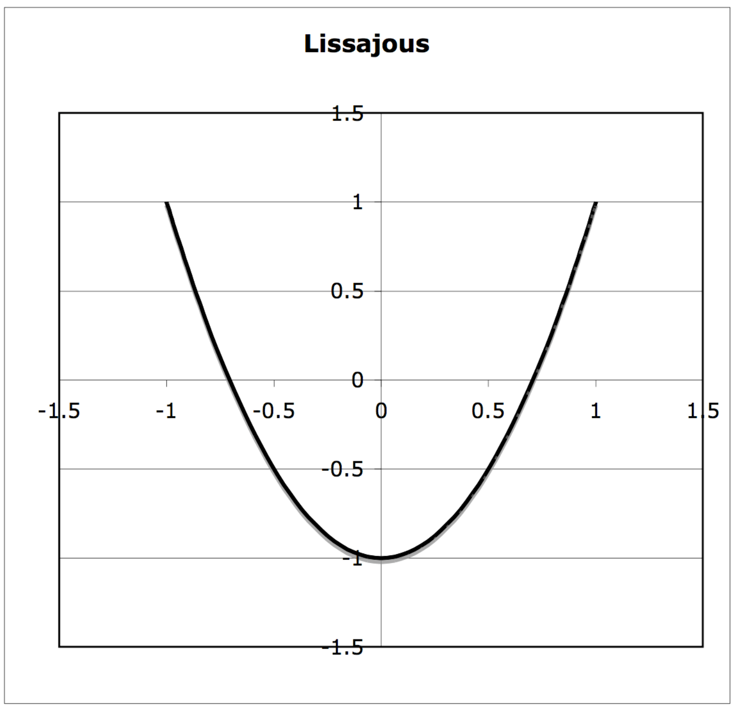


Figure 2.4 – The closed Lissajous trajectory followed by the mass in the $(x,\ y)$-plane. In this case $\omega_y = 2\omega_x$, and $\delta = 0$ ($\alpha = \beta = 0$). Take note of the similarity of this curve to the one shown in Figure 2.3 for $\delta = \pi/2$ ($\alpha = 3\pi/2$ and $\beta = \pi$).

2.1 Damped Oscillations

The oscillatory motions described in the preceding sections are called **free oscillations**, since once set, they would never stop. In physical systems, there will often be dissipative or frictional forces that will dampen the oscillations, and eventually erase them. It is usually assumed that the damping forces are a linear function of the velocity. For the one-dimensional case, the force can be represented by

$$F_d = -b\dot{x}, \tag{2.28}$$

where the parameter $b > 0$. This is a necessary condition to insure that the damping force is indeed resisting motions. Adding this resistive force to the restoring force responsible for the oscillatory motion, we write the Newtonian equation of motion as

$$m\ddot{x} = -b\dot{x} - kx, \tag{2.29}$$

or

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^{\ 2}x = 0}. \tag{2.30}$$

We have introduced the **damping coefficient** $\beta \equiv b/2m$ in equation (2.30), while the frequency of oscillation retains its original definition (i.e., $\omega_0 = \sqrt{k/m}$). The solution to the homogeneous second order differential equation (2.30) is given by

$$\boxed{x(t) = e^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2 - \omega_0^{\ 2}}\,t\right) + A_2 \exp\left(-\sqrt{\beta^2 - \omega_0^{\ 2}}\,t\right)\right]}. \tag{2.31}$$

This is done by evaluating the roots of the characteristic, or auxiliary, equation $r^2 + 2\beta r + \omega_0^{\ 2} = 0$ for equation (2.30), and using them as arguments for the exponential functions in the solution. The roots are

$$\begin{aligned} r_1 &= -\beta + \sqrt{\beta^2 - \omega_0^{\ 2}} \\ r_2 &= -\beta - \sqrt{\beta^2 - \omega_0^{\ 2}}. \end{aligned} \tag{2.32}$$

Underdamped Motion

This is the case when $\omega_0^{\ 2} > \beta^2$. We define a new quantity

$$\omega_1^{\ 2} \equiv \omega_0^{\ 2} - \beta^2, \tag{2.33}$$

which will always be positive for an underdamped oscillator. Because of this, the exponents within the brackets of equation (2.31) are imaginary, and the equation becomes

$$x(t) = e^{-\beta t}\left[A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right]. \tag{2.34}$$

But since $x(t)$ is a real function we must have $A_1^* = A_2$ and therefore

$$\begin{aligned} x(t) &= e^{-\beta t}\frac{A}{2}\left[e^{i(\omega_1 t-\delta)}+e^{-i(\omega_1 t-\delta)}\right] \\ &= Ae^{-\beta t}\cos(\omega_1 t-\delta) \end{aligned} \tag{2.35}$$

where $A_1 = e^{-i\delta} A/2$. It then becomes evident that ω_1 is the angular frequency of the damped oscillator. The amplitude of the motion decreases with time because of the $\exp(-\beta t)$ factor present in equation (2.35).

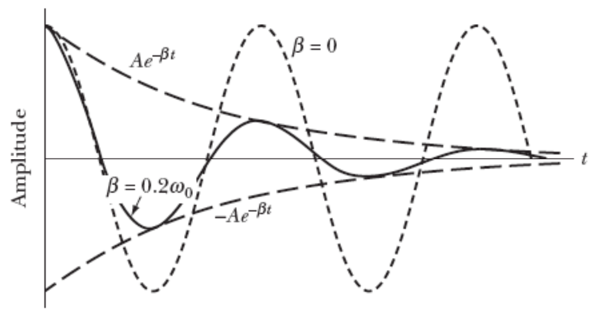


Figure 2.5 – The underdamped motion (solid line) is an oscillatory motion (short dashes) that decreases within the exponential envelope (long dashes).

Critically Damped Motion

When $\beta^2 = \omega_0{}^2$ the roots of the characteristic equation are equal $r_1 = r_2 = -\beta$. The solution to equation (2.30) (the solution (2.31) is not valid for this case) is then

$$(A + Bt)e^{-\beta t}. \tag{2.36}$$

A critically damped system will approach equilibrium at a more rapid rate than it would for any other level of damping.

Overdamped Motion

When $\beta^2 > \omega_0{}^2$, the roots (equation (2.32)) of the characteristic equation are both real. The solution is given by

$$x(t) = e^{-\beta t}\left[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right], \tag{2.37}$$

where

$$\omega_2 = \sqrt{\beta^2 - \omega_0{}^2}. \tag{2.38}$$

There are no oscillatory motions in the case of an overdamped system.

The three cases of damped oscillations (i.e., underdamped, critically damped, and overdamped) are shown in Figure 2.6.

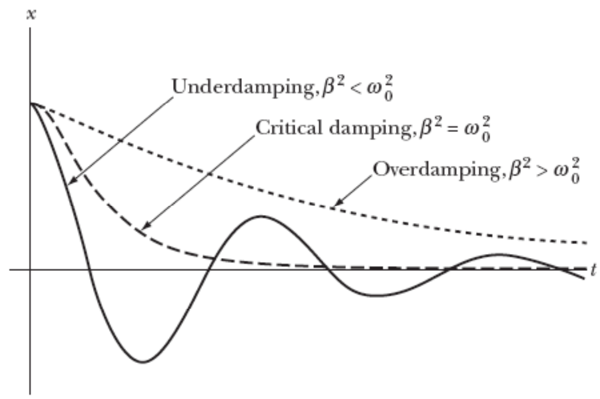


Figure 2.6 – Damped oscillator motions for the three cases of damping.

Problems

(The numbers refer to the problems at the end of Chapter 3 in Thornton and Marion.)

3-11. Derive the expressions for the energy and energy-loss curves shown in Figure 2.7, below, for the underdamped oscillator. For a lightly damped oscillator, calculate the *average rate* at which the underdamped oscillator loses energy (i.e., compute a time average over one cycle).

The total energy of the oscillator is given by

$$E = \frac{1}{2}m\dot{x}(t)^2 + \frac{1}{2}kx(t)^2,$$

where the position and the velocity of the mass as a function of time are given by equation (2.35)

$$\begin{aligned} x(t) &= Ae^{-\beta t}\cos(\omega_1 t - \delta) \\ \dot{x}(t) &= -Ae^{-\beta t}\left[\beta\cos(\omega_1 t - \delta) + \omega_1\sin(\omega_1 t - \delta)\right], \end{aligned}$$

with $\omega_1 = \sqrt{\omega_0^{\,2} - \beta^2}$ and $\omega_0 = \sqrt{k/m}$. We then have for the total energy

$$E = \frac{1}{2}A^2me^{-2\beta t}\left[\beta^2\cos^2(\omega_1 t-\delta)+2\beta\omega_1\sin(\omega_1 t-\delta)\cos(\omega_1 t-\delta)\right.$$
$$\left.+\omega_1^{\,2}\sin^2(\omega_1 t-\delta)+\omega_0^{\,2}\cos^2(\omega_1 t-\delta)\right]$$
$$= \frac{1}{2}A^2me^{-2\beta t}\left[(\beta^2+\omega_0^{\,2})\cos^2(\omega_1 t-\delta)+\omega_1^{\,2}\sin^2(\omega_1 t-\delta)\right.$$
$$\left.+2\beta\omega_1\sin(\omega_1 t-\delta)\cos(\omega_1 t-\delta)\right]$$
$$= \frac{1}{4}A^2me^{-2\beta t}\left[(\beta^2+\omega_0^{\,2}+\omega_1^{\,2})+(\beta^2+\omega_0^{\,2}-\omega_1^{\,2})\cos(2[\omega_1 t-\delta])\right.$$
$$\left.+2\beta\omega_1\sin(2[\omega_1 t-\delta])\right],$$

and finally

$$\boxed{E = \frac{1}{2}A^2me^{-2\beta t}\left[\omega_0^{\,2}+\beta^2\cos(2[\omega_1 t-\delta])+\beta\omega_1\sin(2[\omega_1 t-\delta])\right]}.$$

We take the time derivative of the last equation to find dE/dt

$$\boxed{\begin{aligned}\frac{dE}{dt} = -A^2me^{-2\beta t}\Big[&\beta\omega_0^{\,2}+\beta(2\beta^2-\omega_0^{\,2})\cos(2[\omega_1 t-\delta])\\ &+2\beta^2\omega_1\sin(2[\omega_1 t-\delta])\Big]\end{aligned}}$$

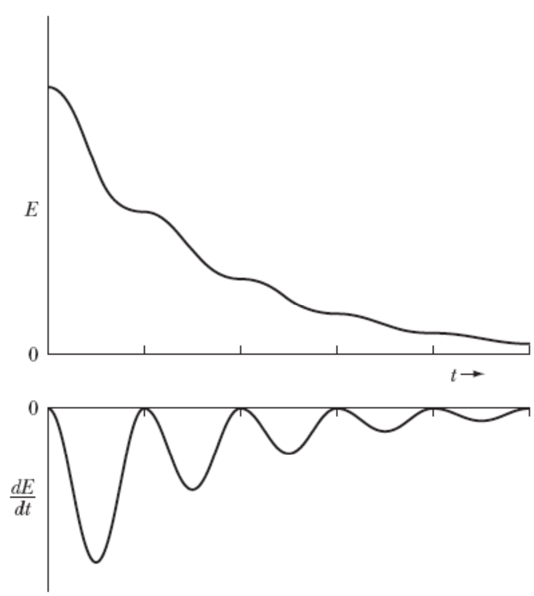
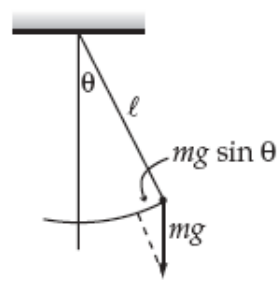


Figure 2.7 – The total energy and rate of energy loss for the underdamped oscillator.

Before evaluating the time average of the energy loss over a cycle, we must realize that for an underdamped oscillator $\beta \ll \omega_0 \simeq \omega_1$. This means that the exponential term $e^{-2\beta t}$ will not vary appreciably in the time it takes for the oscillator to complete one cycle. The sine and cosine term will nearly average to zero, and the only term of importance left is

$$\boxed{\frac{dE}{dt} \simeq -A^2 m\beta\omega_0{}^2 e^{-2\beta t}}.$$

3-12. A simple pendulum consists of a mass *m* suspended from a fixed point by a weightless, extension less rod of length *l*. Obtain the equation of motion and, in the approximation that $\sin(\theta) \cong \theta$, show that the natural frequency is $\omega_0 = \sqrt{g/l}$, where *g* is the gravitational field strength (i.e., the acceleration). Discuss the motion in the event that the motion takes place in a viscous medium with a retarding force $2m\sqrt{gl}\,\dot{\theta}$.



The equation of (angular) motion can simply be written as

$$I\ddot{\theta} + 2ml\sqrt{gl}\ \dot{\theta} + mgl\sin(\theta) = 0$$

or

$$I\ddot{\theta} + 2m\sqrt{gl^3}\,\dot{\theta} + mgl\theta = 0,$$

where $I = ml^2$. We re-write this equation as

$$\ddot{\theta} + 2\beta\dot{\theta} + \omega_0{}^2\theta = 0$$

with $\beta = \sqrt{g/l}$, and $\omega_0 = \sqrt{g/l}$. The natural frequency of the system is defined as that at which the pendulum would oscillate at if $\beta = 0$. It is, therefore, given by ω_0. We can verify the amount of damping by evaluating $\omega_0{}^2 - \beta^2$. It is found to equal to zero. We are, therefore, in the presence of a system where the two roots of the auxiliary equation are equal (see equations (2.32)). That is to say, the system is critically damped. We know that the solution to the second order differential equation that defines the oscillatory motion is

$$\theta(t) = (A + Bt)e^{-\beta t}.$$

If we define the initial conditions as θ_0, and $\dot{\theta}_0 = 0$ (i.e., the pendulum is initially at rest), we find

$$\theta(t) = \theta_0(1 + \beta t)e^{-\beta t}.$$

2.2 Sinusoidal Driving Forces

So far, we only considered cases where there were no external forces driving the oscillator. A simple example consists of introducing a sinusoidal varying force to the problem. In this case, the total force acting on the particle is

$$F = -kx - b\dot{x} + F_0 \cos(\omega t),$$

where F_0 and ω are the amplitude and the angular frequency of the external force, respectively. The equation of motion is then

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0{}^2 x = A\cos(\omega t)} \tag{2.39}$$

where $A = F_0/m$. The solution of equation (2.39) is composed of two parts, the **complementary function** $x_c(t)$, which the solution to the homogeneous equation (2.30), and the **particular equation** $x_p(t)$, which is the system's response to the external driving force reproduced on the right-hand side of equation (2.39). We can, therefore, write

$$x(t) = x_c(t) + x_p(t).$$

The precise form of $x_c(t)$ will depend on the case at hand, i.e., whether the oscillator is under-, critically, or overdamped. The corresponding equation will be given by either equation (2.35), (2.36), or (2.37). For the particular solution, we try

$$x_p(t) = D\cos(\omega t - \delta). \tag{2.40}$$

Inserting equation (2.40) into equation (2.39) we get

$$D\left[\left(-\omega^2+\omega_0{}^2\right)\cos(\omega t-\delta)-2\beta\omega\sin(\omega t-\delta)\right]=A\cos(\omega t). \tag{2.41}$$

But since

$$\begin{aligned}\cos(\omega t-\delta)&=\cos(\delta)\cos(\omega t)+\sin(\delta)\sin(\omega t)\\ \sin(\omega t-\delta)&=\sin(\omega t)\cos(\delta)-\sin(\delta)\cos(\omega t)\end{aligned} \tag{2.42}$$

we can expend equation (2.41) to get

$$\begin{aligned}&\left\{A-D\left[\left(\omega_0{}^2-\omega^2\right)\cos(\delta)+2\beta\omega\sin(\delta)\right]\right\}\cos(\omega t)\\ &\quad -D\left[\left(\omega_0{}^2-\omega^2\right)\sin(\delta)-2\beta\omega\cos(\delta)\right]\sin(\omega t)=0.\end{aligned} \tag{2.43}$$

Since $\sin(\omega t)$ and $\cos(\omega t)$ are linearly independent, equation (2.43) can only be true if the two terms on the left-hand side vanish simultaneously. From the second term we get

$$\tan(\delta)=\frac{2\beta\omega}{\omega_0{}^2-\omega^2}, \tag{2.44}$$

which implies that

$$\begin{aligned}\sin(\delta)&=\frac{2\beta\omega}{\sqrt{\left(\omega_0{}^2-\omega^2\right)^2+4\beta^2\omega^2}}\\ \cos(\delta)&=\frac{\omega_0{}^2-\omega^2}{\sqrt{\left(\omega_0{}^2-\omega^2\right)^2+4\beta^2\omega^2}}.\end{aligned} \tag{2.45}$$

From the first term on the left-hand side of equation (2.43) we find that

$$\begin{aligned}D&=\frac{A}{\left(\omega_0{}^2-\omega^2\right)\cos(\delta)+2\beta\omega\sin(\delta)}\\ &=\frac{A}{\sqrt{\left(\omega_0{}^2-\omega^2\right)^2+4\beta^2\omega^2}}.\end{aligned} \tag{2.46}$$

The particular solution is, therefore, given by

$$\boxed{x_p(t)=\frac{A}{\sqrt{\left(\omega_0{}^2-\omega^2\right)^2+4\beta^2\omega^2}}\cos(\omega t-\delta)} \tag{2.47}$$

with

$$\boxed{\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^{\ 2} - \omega^2}\right)}. \tag{2.48}$$

As was previously mentioned, combining the complementary and the particular equations gives the complete solution to the problem

$$\boxed{x(t) = x_c(t) + x_p(t)}. \tag{2.49}$$

Because of the fact that $x_c(t)$ approaches zero after a long enough amount of time (its envelope scales with $e^{-\beta t}$), it is often called the *transient response*. On the other hand, the particular solution will not cancel out unless the driving force ceases. For this reason, it is called the *permanent* or *steady state response*. Thus,

$$x(t \gg 1/\beta) = x_p(t). \tag{2.50}$$

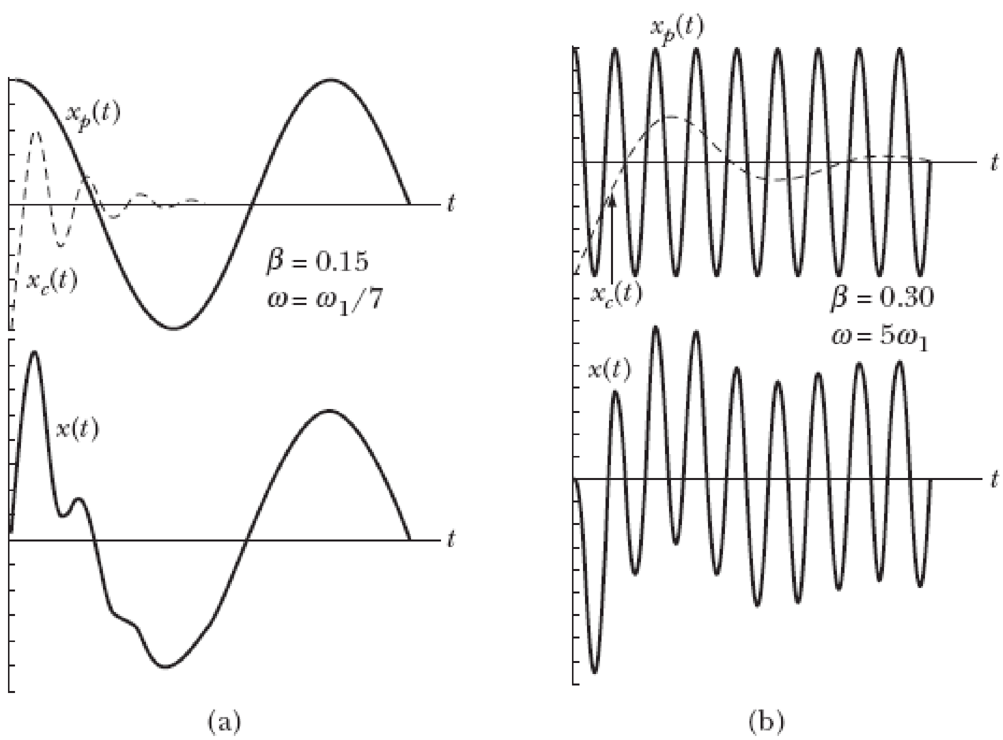


Figure 2.8 – Sinusoidally driven oscillatory motions with damping. The permanent solution $x_p(t)$, the transient solution $x_c(t)$, and the sum of the two $x(t)$ are shown in (a) for driving frequency ω smaller than the oscillation frequency ω_1 ($\omega < \omega_1$) and in (b) for $\omega > \omega_1$.

Resonance Phenomena

The amplitude of the oscillator's response to the driving input will be a function of frequency. To find at what frequency ω_R the amplitude will be at a maximum we calculate

$$\left.\frac{dD}{d\omega}\right|_{\omega=\omega_R} = 0. \tag{2.51}$$

Because of the form of the equation for D obtained earlier (i.e., equation (2.46)), this is equivalent to setting

$$\begin{aligned}\frac{d}{d\omega}\left(\left(\omega_0^{\,2}-\omega^2\right)^2+4\omega^2\beta^2\right)&=0\\&=-4\omega\left(\omega_0^{\,2}-\omega^2\right)+8\omega\beta^2,\end{aligned} \tag{2.52}$$

which gives

$$\omega_R = \sqrt{\omega_0^{\,2}-2\beta^2}. \tag{2.53}$$

It is seen that the resonance frequency is a function of the damping coefficient β. More precisely, more damping will lower the resonant frequency until the point where no resonance can occur when $\beta > \omega_0/\sqrt{2}$.

The degree of damping, or, inversely, the quality of the resonance, is often described in terms of the *quality* or *Q-factor* of the system

$$Q \equiv \frac{\omega_R}{2\beta}. \tag{2.54}$$

In cases where β is small, it can be shown that

$$Q \cong \frac{\omega_0}{\Delta\omega} \tag{2.55}$$

where $\Delta\omega$ represents the frequency interval between the half-power points (i.e., where the amplitude is $1/\sqrt{2}$ of its maximum). The higher the Q of the system, the more the energy from the driving input is stored in the oscillator, and less is dissipated. Figure 2.9, below, shows the amplitude D and the phase δ of the response of an oscillator as a function of the driving frequency, for different values of Q's.

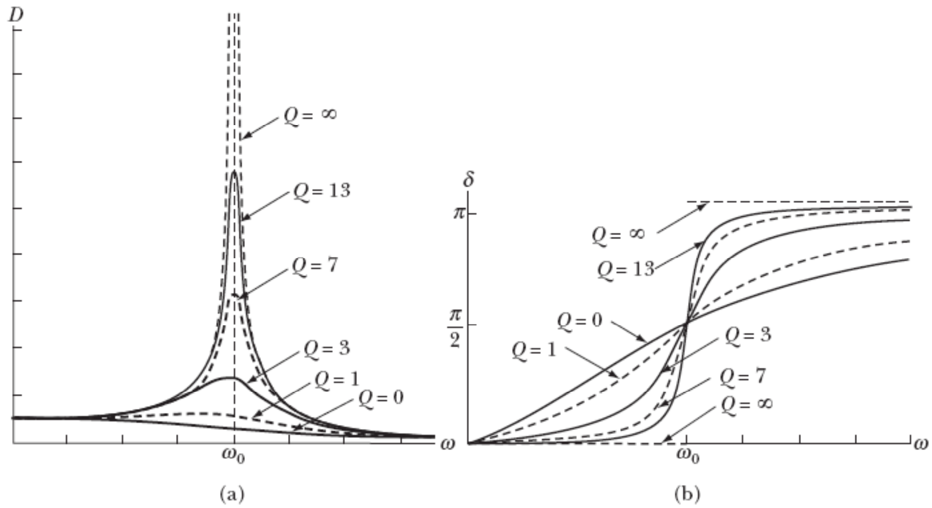


Figure 2.9 – (a) The amplitude D and (b) the phase δ of the response of an oscillator as a function of the driving frequency, for different values of Q.

2.3 Arbitrary Driving Forces

Periodic signals

Whenever the driving force is periodical (i.e., $F(t+\tau)=F(t)$, with τ the period), it can always be expanded in a Fourier series

$$F(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left[a_n\cos(n\omega t)+b_n\sin(n\omega t)\right] \tag{2.56}$$

with

$$\begin{aligned} a_n &= \frac{2}{\tau}\int_0^{\tau}F(t')\cos(n\omega t')\,dt' \\ b_n &= \frac{2}{\tau}\int_0^{\tau}F(t')\sin(n\omega t')\,dt' \end{aligned} \tag{2.57}$$

and $\omega=2\pi/\tau$. Alternatively, equation (2.56) can be written in a slightly different form

$$F(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}c_n\cos(n\omega t-\phi_n) \tag{2.58}$$

with

$$c_n = \sqrt{a_n^2 + b_n^2}$$
$$\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right). \tag{2.59}$$

The equation of motion in this case is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F(t)}{m}. \tag{2.60}$$

Since this equation (and the system) is linear, the solution will be once again expressed as

$$x(t) = x_c(t) + x_p(t), \tag{2.61}$$

but if $x_c(t)$ is still given by (2.35), (2.36), or (2.37) (depending on the damping level), the steady state $x_p(t)$ solution takes, using equations (2.47) and (2.48), the form of a series

$$x_p(t) = \frac{1}{m}\left[\frac{a_0}{2\omega_0^2} + \sum_{n=1}^{\infty} \frac{c_n}{\sqrt{\left(\omega_0^2 - n^2\omega^2\right) + 4\beta^2 n^2 \omega^2}} \cos\left(n\omega t - \phi_n - \delta_n\right)\right] \tag{2.62}$$

with

$$\delta_n = \tan^{-1}\left(\frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}\right). \tag{2.63}$$

The first term on the right-hand side of equation (2.62) represents the oscillator's response to the constant component of the driving force, as can be simply verified from equation (2.60).

Arbitrary signals

(The material covered in this section is optional, and is not subject to examination. It will not be found in Thornton and Marion. Parts of it can, however, be found in an earlier edition of the same book (i.e., J.B. Marion, Classical Dynamics of Particles and Systems, 2nd edition (Academic Press), pp 142-145)

In the preceding sections we have mainly used straightforward methods for solving differential equations, by mostly guessing at the particular form of the solution for the problem at hand. There exists, however, different techniques that allow for a more systematic treatment of linear differential equations. One particularly powerful method utilizes the **Laplace transform**. Its strength lies in the fact that it allows the transformation of a *differential* equation to an *algebraic* equation. This is a method of choice to deal with more complicated functions.

The Laplace transform is defined as follows

$$X(s) \equiv L\left[x(t)\right] = \int_0^\infty x(t) e^{-st} dt, \tag{2.64}$$

where the variable s is defined as containing both a real and imaginary part, i.e., $s = \sigma + i\omega$ with $\sigma \geq 0$ such that e^{-st} remains finite as $t \to \infty$. Referring to equation (2.64), we say that "$X(s)$ is the Laplace transform of $x(t)$". We also assumed that the time variable t starts at 0, but this could be changed to any other value (e.g., t_0).

For example, we calculate the Laplace transform of a few simple functions

$$\begin{aligned} L[A] &= \int_0^\infty A e^{-st} dt = -\frac{A}{s} e^{-st}\Big|_0^\infty = \frac{A}{s}, && s > 0 \\ L\left[e^{-at}\right] &= \int_0^\infty e^{-at} e^{-st} dt = -\frac{1}{s+a} e^{-(s+a)t}\Big|_0^\infty = \frac{1}{s+a}, && s > -a. \end{aligned} \tag{2.65}$$

A particularly important transform is that of an impulse of time duration τ defined as

$$\begin{aligned} x(t) &= \frac{1}{\tau}, && 0 < t < \tau \\ &= 0, && |t| > \tau \end{aligned} \tag{2.66}$$

$$X(s) = \frac{1}{\tau} \int_0^\tau e^{-st} dt = -\frac{1}{s\tau} e^{-st}\Big|_0^\tau = \frac{1}{s\tau}\left(1 - e^{-s\tau}\right). \tag{2.67}$$

If we now take the limit of equation (2.67) when $\tau \to 0$, we get

$$\lim_{\tau \to 0} X(s) = \frac{1}{s\tau}\left(1 - \left[1 - s\tau\right]\right) = 1. \tag{2.68}$$

The limit of a function such as $x(t)$, defined in equation (2.66), when the duration of the impulse is taken to be infinitely small while keeping the area of the impulse constant is called a **Dirac** or **delta function**. It is usually simply written as $\delta(t)$, and has the property that $\delta(t) = 0$ for $t \neq 0$ and $\delta(t) = \infty$ for $t = 0$, but

$$\int_{-\infty}^\infty \delta(t - t_0) dt = 1. \tag{2.69}$$

Equally important is the transform of a **step** or **Heaviside function**, represented by

$$\begin{aligned} u_{-1}(t) &= 1, \qquad t > 0 \\ &= 0, \qquad t < 0. \end{aligned} \tag{2.70}$$

Since our definition of the Laplace transform truncates any functions that are non-zero for $t < 0$, the Laplace transform of the step function was evaluated in equation (2.65) and found to be

$$L\left[u_{-1}(t)\right] = \frac{1}{s}. \tag{2.71}$$

The list of transforms appearing in Table 2.1 can be similarly verified.

Table 2.1 – Laplace transform pairs

$L\left[A\delta(t)\right] = A,$	$s > 0$
$L\left[Au_{-1}(t)\right] = \dfrac{A}{s},$	$s > 0$
$L\left[e^{-at}\right] = \dfrac{1}{s+a},$	$s > -a$
$L\left[t^n\right] = \dfrac{n!}{s^{n+1}},$	$s > 0, \quad n = 1, 2, 3, \ldots$
$L\left[t^n e^{-at}\right] = \dfrac{n!}{(s+a)^{n+1}},$	$s > -a, \quad n = 1, 2, 3, \ldots$
$L\left[\sin(\omega t)\right] = \dfrac{\omega}{s^2+\omega^2},$	$s > 0$
$L\left[\cos(\omega t)\right] = \dfrac{s}{s^2+\omega^2},$	$s > 0$
$L\left[e^{-at}\sin(\omega t)\right] = \dfrac{\omega}{(s+a)^2+\omega^2},$	$s > -a$
$L\left[e^{-at}\cos(\omega t)\right] = \dfrac{s+a}{(s+a)^2+\omega^2},$	$s > -a$

The Laplace transform also possesses other important properties, some of which are

I. Linearity. If A and B are constants

$$L\left[Ax(t) + By(t)\right] = AX(s) + BY(s) \tag{2.72}$$

II. Transform of derivatives.

$$L\left[\frac{dx(t)}{dt}\right] = \int_0^\infty \frac{dx(t)}{dt} e^{-st} dt = x(t)e^{-st}\Big|_0^\infty + s\int_0^\infty x(t)e^{-st} dt$$
$$= sX(s) - x(0) \tag{2.73}$$

where we integrated by parts, and $x(0)$ is the initial condition $x(t)$. Similarly, the transform of higher derivatives can be shown to give

$$L\left[\frac{d^2x(t)}{dt^2}\right] = s^2 - sx(0) - \frac{dx(t)}{dt}\bigg|_{t=0}$$
$$L\left[\frac{d^nx(t)}{dt^n}\right] = s^n - \sum_{k=1}^{n} s^{n-k} \frac{d^{k-1}x(t)}{dt^{k-1}}\bigg|_{t=0} \tag{2.74}$$

III. Transform of primitive of functions.

$$L\left[\int_0^t x(\tau)d\tau\right] = \int_0^\infty \left\{\int_0^t x(\tau)d\tau\right\} e^{-st} dt$$
$$= -\frac{1}{s}\left\{\int_0^t x(\tau)d\tau\right\} e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty x(t)e^{-st} dt \tag{2.75}$$
$$= \frac{X(s)}{s} + \frac{1}{s}\left\{\int_0^t x(\tau)d\tau\right\}\bigg|_{t=0}$$

where we again integrated by parts. The transform of higher primitives is given by

$$L\left[\int \cdots \int x(\tau)(d\tau)^n\right] = \frac{X(s)}{s^n} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}}\left\{\int \cdots \int x(\tau)(d\tau)^k\right\}\bigg|_{t=0} \tag{2.76}$$

IV. Time shifting. Since $x(t) = 0$ for $t < 0$, we can write

$$L\left[x(t-\tau)\right] = \int_0^\infty x(t-\tau)e^{-st} dt = \int_\tau^\infty x(t-\tau)e^{-st} dt$$
$$= \int_0^\infty x(\lambda)e^{-s(\lambda+\tau)} d\lambda = e^{-s\tau}\int_0^\infty x(\lambda)e^{-s\lambda} d\lambda \tag{2.77}$$
$$= e^{-s\tau}X(s)$$

where we made the substitution $\lambda = t - \tau$.

V. Multiplication by an exponential.

$$L\left[e^{-at}x(t)\right] = \int_0^\infty e^{-at}x(t)e^{-st}dt = \int_0^\infty x(t)e^{-(s+a)t}dt \\ = X(s+a) \tag{2.78}$$

The residue theorem

Once a function or an equation has been transformed in the Laplace domain, then modified for one purpose or another, it will eventually need to be transformed back to the time domain. Although an inverse Laplace transform can be mathematically defined, it is always more convenient and easier to use the so-called *residue theorem* to go from the Laplace to the time domain. This theorem is stated as follows. Given a function $X(s)$, for which the denominator can be written as a product of factors of the type $(s+a_j)^m$ (where a_j is called a *pole* or order *m*), we can write

$$x(t) = L^{-1}\left[X(s)\right] \\ = \sum_{j=1}^{n} \frac{1}{(m-1)!} \lim_{s\to -a_j}\left(\frac{d^{m-1}}{ds^{m-1}}\left[\left(s+a_j\right)^m X(s)e^{st}\right]\right), \quad t>0 \tag{2.79}$$

where *n* is the number of poles in the denominator of $X(s)$, and the quantity in between the curly braces is called the residue of $X(s)e^{st}$ at the pole a_j of order *m*. Let's consider a few examples

$$x(t) = L^{-1}\left[\frac{1}{s+a}\right] = \lim_{s\to -a}\frac{1}{0!}\frac{d^0}{ds^0}\left[(s+a)\cdot\frac{e^{st}}{s+a}\right] \\ = e^{-at}, \qquad t>0 \\ x(t) = L^{-1}\left[\frac{1}{(s+a)^2}\right] \\ = \lim_{s\to -a}\frac{1}{1!}\frac{d^1}{ds^1}\left[(s+a)^2\cdot\frac{e^{st}}{(s+a)^2}\right] \\ = te^{-at}, \qquad t>0 \tag{2.80}$$

and finally

$$
\begin{aligned}
x(t) &= L^{-1}\left[\frac{s+a}{(s+a)^2+\omega^2}\right] = L^{-1}\left[\frac{s+a}{\left(s+(a-i\omega)\right)\left(s+(a+i\omega)\right)}\right] \\
&= \lim_{s\to -a+i\omega}\frac{1}{0!}\frac{d^0}{ds^0}\left[\left(s+(a-i\omega)\right)\frac{(s+a)}{(s+a)^2+\omega^2}e^{st}\right] \\
&\quad + \lim_{s\to -a-i\omega}\frac{1}{0!}\frac{d^0}{ds^0}\left[\left(s+(a+i\omega)\right)\frac{(s+a)}{(s+a)^2+\omega^2}e^{st}\right] \\
&= \frac{i\omega e^{(-a+i\omega)t}}{2i\omega} + \frac{-i\omega e^{(-a-i\omega)t}}{-2i\omega} \\
&= e^{-at}\cos(\omega t), \qquad t>0.
\end{aligned}
\tag{2.81}
$$

These results can be verified against the examples presented in Table 2.1.

Application to the damped oscillator problem

Let's now solve a few cases involving the equation of motion of a damped oscillator with different types of driving input. The equation to solve is

$$
\boxed{\ddot{x}(t)+2\beta\dot{x}(t)+\omega_0^{\,2}x(t)=f(t)}
\tag{2.82}
$$

I. $f(t)=A\delta(t)$.

$$
L\left[\ddot{x}(t)+2\beta\dot{x}(t)+\omega_0^{\,2}x(t)\right]=L\left[f(t)\right]
\tag{2.83}
$$

Using the linearity property of the Laplace transform and Table 2.1, we get

$$
\begin{aligned}
&\left(s^2X(s)-sx_0-\dot{x}_0\right)+2\beta\left(sX(s)-x_0\right)+\omega_0^{\,2}X(s)=A \\
&X(s)\left(s^2+2\beta s+\omega_0^{\,2}\right)=A+x_0\left(s+2\beta\right)+\dot{x}_0
\end{aligned}
\tag{2.84}
$$

In everything that will follow, we will assume that $x_0=\dot{x}_0=0$. We now solve equation (2.84)

$$
\begin{aligned}
X(s) &= \frac{A}{s^2+2\beta s+\omega_0^{\,2}} \\
&= \frac{A}{\left(s+\left(\beta-\sqrt{\beta^2-\omega_0^{\,2}}\right)\right)\left(s+\left(\beta+\sqrt{\beta^2-\omega_0^{\,2}}\right)\right)}
\end{aligned}
\tag{2.85}
$$

We now use the residue theorem stated in equation (2.79)

$$x(t) = A\left[\frac{e^{-\left(\beta-\sqrt{\beta^2-\omega_0{}^2}\right)t}}{2\sqrt{\beta^2-\omega_0{}^2}} - \frac{e^{-\left(\beta+\sqrt{\beta^2-\omega_0{}^2}\right)t}}{2\sqrt{\beta^2-\omega_0{}^2}}\right]$$
$$= A\frac{e^{-\beta t}}{2\sqrt{\beta^2-\omega_0{}^2}}\left[e^{\sqrt{\beta^2-\omega_0{}^2}\,t} - e^{-\sqrt{\beta^2-\omega_0{}^2}\,t}\right], \qquad t > 0 \tag{2.86}$$

A close examination of equation (2.86) shows that the response of the damped oscillator to a Dirac function is nothing more than the complementary solution of the equation of motion (compare with equation (2.31)). In the case of the underdamped oscillator ($\beta^2 < \omega_0{}^2$), we find that

$$x(t) = A\frac{e^{-\beta t}}{\omega_1}\sin(\omega_1 t), \qquad t > 0 \tag{2.87}$$

with $\omega_1 = \sqrt{\omega_0{}^2 - \beta^2}$.

II. $f(t) = Au_{-1}(t)$

In this case, we have (assuming that $\beta^2 < \omega_0{}^2$, and $\omega_1 = \sqrt{\omega_0{}^2 - \beta^2}$)

$$X(s)\left(s^2 + 2\beta s + \omega_0{}^2\right) = \frac{A}{s} \tag{2.88}$$

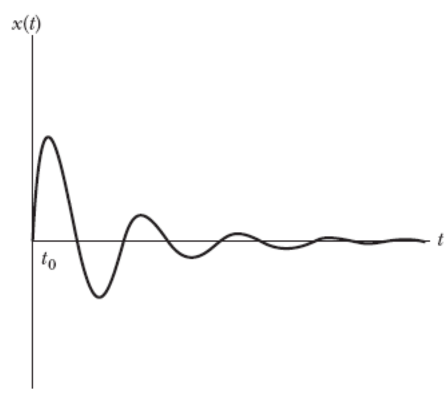


Figure 2.10 – Response to a Dirac function driving input.

$$
\begin{aligned}
x(t) &= \frac{A}{s\left(s^2 + 2\beta s + \omega_0^{\ 2}\right)} \\
&= \frac{A}{s\left(s + (\beta - i\omega_1)\right)\left(s + (\beta + i\omega_1)\right)} \\
&= A\left[\frac{1}{\omega_0^{\ 2}} + \frac{e^{-(\beta - i\omega_1)t}}{2i\omega_1\left(i\omega_1 - \beta\right)} + \frac{e^{-(\beta + i\omega_1)t}}{2i\omega_1\left(i\omega_1 + \beta\right)}\right] \\
&= A\left[\frac{1}{\omega_0^{\ 2}} - \frac{e^{-\beta t}}{\omega_1\sqrt{\beta^2 + \omega_1^{\ 2}}}\cos\left(\omega_1 t - \phi\right)\right], \qquad t > 0
\end{aligned}
\tag{2.89}
$$

with

$$
\phi = \tan^{-1}\left(\frac{\beta}{\omega_1}\right). \tag{2.90}
$$

The Laplace transform can be systematically applied to more complicated types of problems and driving functions (periodic or not). It is also important to realize that the solution to a given problem provided by the application of the Laplace transform **includes both the complementary and the particular solutions**.

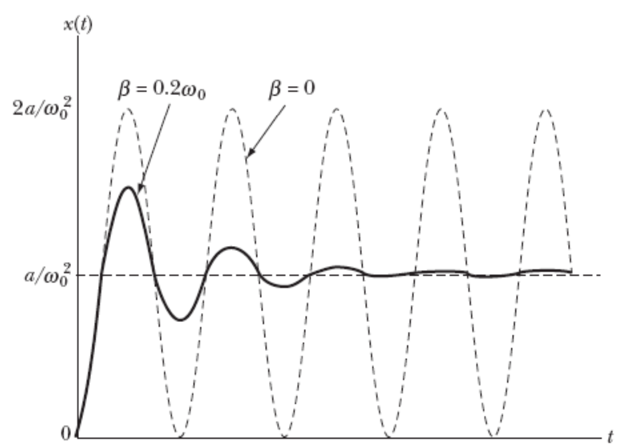


Figure 2.11 – Response to a step function as driving input.